Preface

This is the second book of the trilogy on a fixed income valuation course by Wiley finance. This trilogy covers the following three areas of fixed income valuation:

1. Interest rate risk modeling
2. Term structure modeling
3. Credit risk modeling

Unlike other books in fixed income valuation, which are either too rigorous but mathematically demanding or easy-to-read but lacking in important details, our goal is to provide readability with sufficient rigor. In the first book, we gave a basic introduction to various fixed income securities and their derivatives. The principal focus of the first book was on measuring and managing interest rate risk arising from general nonparallel rate changes in the term structure of interest rates. Due to smoothness in the shapes of term structures, interest rate risk of straight bonds can be managed even without a proper valuation model simply by using empirical duration vectors or empirical key rate duration profiles. In fact, as demonstrated in the first book, *Interest Rate Risk Modeling, basic* interest rate risk management of financial institutions such as commercial banks, fixed income funds, insurance companies, and pension funds can be done without using the most sophisticated valuation models.

In this second book, we shift our focus to *valuation of fixed income securities and their derivatives*. A good valuation model is a key input not only for supporting the fixed income trading desks of financial institutions, but also for designing comprehensive risk management strategies for these institutions. The basic tools of fixed income valuation are "dynamic term structure models," which are also loosely referred to as "interest rate models." This second book in the trilogy aims to bridge the gap between the advanced technical books in the term structure area and the many elementary discussions of term structure models in the general fixed income trade books. By choosing basic mathematical rules with heuristic derivations over rigorous theoretical developments with technical proofs, we hope to make this difficult subject matter accessible to the wider audience of financial analysts, students, and academics. The readers will also benefit from Excel/VBA–based software, which runs C and C++ programs at the back-end, for valuing securities using the various term structure models given in this book.

Valuation of fixed income securities and their derivatives is of interest to many economic participants, including regular corporations funded with liabilities with embedded options, insurance companies and pension funds with fixed income assets and liabilities, investment banks dealing in swaps and other exotic interest rate products, hedge funds with long and short positions in credit derivatives, savings and commercial banks with prepayment options embedded in their mortgage portfolios, and many such institutions. The purpose of this book is to introduce a broad variety of term structure models, including the affine models, the quadratic models, the Heath, Jarrow, and Morton (HJM) models, and the LIBOR Market Model (LMM) and to show how to price basic interest rate and credit derivative products, such as Treasury futures, Eurodollar futures, bond options, forward rate agreements, interest rate swaps, interest rate caps, interest rate swaptions, credit default swaps, credit spread options, and the like, using some of these models.

UNIQUE FEATURES

This book distinguishes itself from other books in this area by making the following unique contributions:

A Comprehensive Classification Scheme for Term Structure Models

This book classifies all term structure models (TSMs) into four types:

1. Fundamental TSMs
2. Preference-Free Single-Plus TSMs
3. Preference-Free Double-Plus TSMs
4. Preference-Free Triple-Plus TSMs

All fundamental models (such as Vasicek [1977] and Cox, Ingersoll, and Ross (CIR) [1985]) assume a time-homogeneous short rate process and give explicit specifications of the market prices of risks. In contrast to fundamental models, preference-free models *do not require explicit specifications of the market prices of risks* for valuing bonds and interest rate derivatives. Hence, valuation can be done without knowing the risk preferences of the market participants under preference-free models. This book considers three types of preference-free TSMs, given as single-plus, double-plus, and triple-plus models. The risk-neutral stochastic processes of the state variables under any preference-free single-plus TSM are identical *in form* to the risk-neutral stochastic processes of the state variables under the corresponding fundamental TSM. However, the empirical estimates of the risk-neutral parameters are generally different under these two types of models, since the fundamental model imposes restrictive functional forms on the specifications of MPRs, while the corresponding single-plus model does not. The trick to deriving a single-plus TSM corresponding to a given fundamental TSM is to specify the stochastic bond price process exogenously using the same form of volatility function used under the given fundamental model. The exogenous stochastic bond price process is then combined with an exogenously given solution of the time 0 bond prices or forward rates, which leads to a time-homogeneous risk-neutral short rate process. This trick is explicitly demonstrated in Chapters 4 and 7. The introduction of single-plus TSMs in this book may appeal to both the econometrically inclined academics (who do not generally like time-inhomogeneous processes) and calibration-desiring practitioners (who must always "fit" a model to market prices!).

The preference-free double-plus TSMs are different from the corresponding fundamental TSMs in two ways. Not only are these models free of the market price of risk (MPR) specifications—similar to the single-plus models—but they also allow the model bond prices to exactly fit the initially observed bond prices. Unlike the single-plus TSMs that may require multiple factors to match the model prices with the observed prices, the double-plus TSMs can allow an exact fit even using a single factor. The initially observed bond prices are used as an input under the double-plus TSMs. These models exactly fit the initially observed bond prices by allowing time-inhomogeneity in the drift of the risk-neutral short rate process. Examples of double-plus TSMs include the models by Ho and Lee [1986], Hull and White [1990],[1] Heath, Jarrow, and Morton (HJM) [1992], and Brigo and Mercurio [2001].[2] This book derives double-plus TSMs corresponding to a variety of fundamental affine and quadratic models, in addition to those already given in the literature.

The preference-free triple-plus TSMs are different from the corresponding fundamental TSMs in three ways. Unlike the fundamental models, but similar to single-plus and double-plus models, these models are free of the MPR specifications. Unlike the fundamental and single-plus models but similar to double-plus models, these models allow an exact fit with the initially observed bond prices. However, unlike the fundamental, single-plus, and double-plus models, which all require a *time-homogeneous* specification of volatilities, the triple-plus TSMs allow time-inhomogeneous volatilities (i.e., time-inhomogeneous short rate volatility and/or time-inhomogeneous forward rate volatilities). Examples of triple-plus TSMs include extensions of the models of Hull and White [1990],[3] Black, Derman, and Toy [1990], and Black and Karasinski [1991] with time-inhomogeneous volatilities, and versions of the LIBOR market model with time-inhomogeneous volatilities (see Brigo and Mercurio [2001, 2006] and Rebonato [2002]).

The triple-plus models require a high number of parameters to obtain an exact fit with the chosen plain vanilla derivative instruments and may suffer from the criticism of "smoothing."

A detailed introduction to the four types of term structure models is given in the last section of Chapter 3, and various chapters explicitly consider fundamental models, single-plus models, and double-plus models. We generally do not consider triple-plus models in this book, except for the triple-plus extension of the Vasicek [1977] model in Chapter 4 and the triple-plus versions of the LIBOR market model in Chapter 12. The symbols +, ++, and +++ are used as suffixes after the name of the given fundamental model to denote the single-plus, double-plus, and triple-plus extensions, respectively, throughout the book.

Efficient Recombining Trees for Short Rate Models with State-Dependent Volatility, Stochastic Volatility, and Jumps

Chapter 6 extends the Nelson and Ramaswamy (NR)[1990] transform for generating efficient *recombining* trees under the square root model of Cox, Ingersoll, and Ross. As demonstrated by Nawalkha and Beliaeva [2006], the equation for the movement of the short rate, when the short rate hits the zero boundary, has an error in the original transform proposed by NR. Nawalkha and Beliaeva (NB) correct this error and also truncate the NR tree at the zero boundary. NB show that using the NR tree keeps the short rate tree stuck at zero with a much higher probability than the CIR model requires, and as a result, the bond prices using the NR tree are significantly higher than the prices obtained from the closed-form solution (as reported in Table 2 of NR [1990]). NB's transform, based on a truncated tree, corrects these errors and obtains bond price approximations, which quickly converge to the closed-form solution with much fewer nodes.

Chapter 5 and 6 also demonstrate how to build recombining trees for *jump-extended* models of Vasicek and CIR, with both exponential and lognormal jumps for the jump-size distribution.

Further, building on the results in Nawalkha and Beliaeva [2006], Nawalkha and Beliaeva [2007] introduce a two-dimensional transform for generating recombining trees for *stochastic-volatility-jump (SVJ)* models. Chapter 8 shows how to apply the two-dimensional transform to generate a recombining tree for the short rate under the stochastic volatility–based maximal model in the $A_1(2)$ subfamily.

The Fourier Inversion Method for Valuing Interest Rate Caps

Chapter 5 introduces the *Fourier inversion method* in the context of pricing interest rate caps (or portfolios of options on zero-coupon bonds) under the exponential jump-extended Vasicek model and its preference-free extensions. Chapter 9 extends the Fourier inversion method to value interest rate caps under the multifactor affine models and their preference-free extensions. Since these models nest most of the widely used one-, two-, and three-factor affine models (see Table 9.3), the solutions given in this chapter apply to all of these nested models. An advantage of the Fourier inversion method is that it only uses a single summation to obtain solutions even for models with multiple factors. Solutions under multiple-factor models typically require multiple-order summations using the traditional methods, such as Longstaff and Schwartz [1992]. Chapter 10 extends the Fourier inversion method to value interest rate caps under the multifactor quadratic models, and Chapter 12 uses this method to value interest caps under the LIBOR model extended with stochastic volatility and jumps.

The Cumulant Expansion Method for Valuing Swaptions

Chapter 9 demonstrates how to apply the *cumulant expansion method* of Collin-Dufresne and Goldstein [2001b] to price swaptions (or options on coupon bonds) under the simple $A_M(N)$ models and their preference-free extensions. This approach exploits the fact that the moments of a coupon bond also have affine closed-form solutions. Using these moments to uniquely identify the cumulants of the distribution of the coupon bond, the probability distribution of the coupon bond's future price at the option expiration date is obtained using an Edgeworth expansion technique. This approach is very fast, since no numerical integrations are performed. Our simulations find this approach to be faster than even the Fourier inversion approach, and hence, can be used to price options on zero-coupon bonds or caplets, as well. Chapter 10 generalizes the cumulant expansion method to multifactor quadratic models.

Analytical Solutions of Other Plain Vanilla Derivatives under Multifactor Affine and Quadratic Models

A variety of interest rate derivatives and credit derivatives can be priced using the analytical results and efficient trees given in this book. In addition to caps and swaptions (and related instruments), other plain vanilla derivatives such as Eurodollar futures and credit default swaps (CDS) are of significant interest to fixed income practitioners. We give solutions to both these derivatives under a variety of multifactor affine and quadratic models and their preference-free extensions. The solutions to Eurodollar futures are given with respect to virtually every fundamental and preference-free affine/quadratic model given in Chapters 4 though 10. The CDS solutions are given under multifactor affine models in Chapter 9 and multifactor quadratic models in Chapter 10. The CDS solutions in Chapter 9 nest the formulas of Longstaff, Mithal, and Neis [2003] and Pan and Singleton [2005] using a general multifactor affine model.

Introduction of Multifactor Explosive Square-Root Models

This book introduces *explosive* square-root processes for pricing interest rate and credit derivatives. Though all state variables should have stationary physical processes consistent with positive speeds of mean reversion, it is not economically unreasonable to expect some state variables to have *explosive risk-neutral processes* consistent with negative risk-neutral speeds of mean reversion. We show that if the product of the risk-neutral speed of mean reversion and the risk-neutral long-term mean of a given state variable remain positive, then the entire analytical apparatus of the square-root models remains valid, regardless of whether the risk-neutral speed of mean reversion is positive or negative. As a general rule, we find that square-root models do not allow sufficient volatility for the long-term forward rates. In order to add volatility at the longer end, lower risk-neutral speeds of mean reversion, including negative speeds, are required for some state variables. Curiously, the infinite-maturity forward rate remains constant over any finite interval of time, satisfying the arbitrage condition given in Dybvig, Ingersoll, and Ross [1996], even when one or more square-root state variables follow explosive risk-neutral processes. In the context of pricing credit default swaps, Pan and Singleton [2005] find default intensity to follow an explosive risk-neutral square-root process.

A Detailed Description of a USV Jump-Based LIBOR Model for Valuing Caps

The USV models arise more naturally under the forward rate models of Heath, Jarrow, and Morton (HJM) [1992] and the LIBOR market model (LMM). Chapter 12 gives a detailed description of the Jarrow, Li, and Zhao (JLZ) [2007] model for pricing interest rate caps. The JLZ model extends the LMM by adding USV processes and a jump process. JLZ note that a *symmetric* smile can be produced by the USV extension of the LFM model. In order to generate the asymmetric "hockey-stick shaped" smile, JLZ add a jump component that generates significantly negative jumps in the forward rates. A potential limitation of the JLZ model is that the term structure of risk premiums related to volatility and jump risks are *time-inhomogeneous*. Nawalkha, Beliaeva, and Soto (NBS) [2007b] extend the JLZ model by allowing the USV and jump parameters to be piecewise constant functions of the time to maturity of the forward rate under the respective forward measure. Using piecewise constant functions allows a time-homogeneous specification of the risk premiums.

Basic Software for Affine, Quadratic, and LIBOR Market Models

Similar to the first book in the trilogy, this book comes with software in a user-friendly Excel/VBA format, which covers a variety of models given in the 12 chapters of the book. The CD-ROM accompanying this book includes various Excel/VBA spreadsheets that allow the reader to explore a variety of term structure models given in the book. The software allows valuation of interest rate derivatives by building interest rate trees for low-dimensional affine models, as well as computing solutions using quasi-analytical formulas for higher-dimensional affine, quadratic, and LIBOR market models. Though most of the programs require coding in advanced scientific languages, such as C or C++, the final output is always presented in user-friendly Excel/VBA spreadsheets. These spreadsheets allow readers with basic Excel skills to instantly play with a variety of term structure models to price caps, swaptions, and other interest rate derivatives and credit derivatives.

Self-Empowering the Target Audience

This book is aimed both at the fixed income practitioners, as well as the graduate students of degree programs in mathematical finance, financial engineering, and MBA/MS/PhD in finance. The writing style of this book has been deliberately chosen to self-empower readers who wish to develop quantitative skills required for valuation of fixed income securities and their derivatives. Though the new unifying framework presented in this book will appeal to the seasoned academics as well, it will be especially insightly for students and financial analysts who are new to this field. As in the first book of the trilogy, we expect that after reading chapters on given topics from the curent book, the reader should be able to follow the examples and be ready to apply these models. Since this books is part of the trilogy, it is integrated both conceptually and in terms of the mathematical notation with the next book, *Credit Risk Modeling*, which covers both the structural and reduced-form models for valuing credit derivatives.

As part of our mission of *Self Empowerment*, thirty percent of the royalties from this book will be donated to The Art of Living Foundation (ww.ArtOfLiving.org), which provide highly effective workshops for stress reduction and increased productivity for busy urban professionals (www.apexcourse.org).

Various aspects of this trilogy on the fixed income valuation course including the book descriptions, software details, online training seminars, and future updates are available on the website www.fixedincomerisk.com

CHAPTER CONTENTS

Chapter 1

This chapter introduces continuous-time diffusion and mixed jump-diffusion processes using heuristic derivations that serve to strengthen the mathematical intuition of the readers. The chapter gives some of the widely applied results from stochastic calculus, a field of mathematics that combines calculus with probability, and shows how these results can be used for modeling the term structure dynamics. The examples in this chapter highlight the mathematical intuition without worrying too much about the "regularity" conditions underlying the continuous-time framework. The results in this chapter may be skipped by readers who are well versed in continuous-time mathematics, but they could be helpful to readers unfamiliar with this branch of mathematics.

Chapter 2

The first half of this chapter gives an intuitive description of the martingale valuation theory using an example of a two-period discrete information structure. Using this example, the chapter gives a heuristic demonstration of the celebrated result that "absence of arbitrage guarantees the existence of an *equivalent* martingale measure under which discounted prices are martingales." Different types of martingale measures, such as the risk-neutral measure and the forward measure, are identified using different numeraires. The chapter also introduces other important concepts related to martingale valuation theory, such as the stochastic discount factor, the Radon-Nikodym derivative, and the Feynman-Kac theorem, in an intuitive manner. The continuous-time counterparts to the discrete-time results are given in the second half of the chapter.

Chapter 3

This chapter introduces basic pricing frameworks for valuing interest rate and credit derivatives, including futures on time deposits (e.g., Eurodollar and Euribor futures), bond futures (e.g., T-bill, T-note, and T-bond futures), bond options, forward rate agreements, interest rate swaps, interest rate options (e.g., caps, floors, and collars), swaptions, and credit default swaps. The chapter describes important features of these derivatives and highlights the underlying relationships among derivative prices. The final section of this chapter introduces a new taxonomy for term structure models that classifies all models as either fundamental models or preference-free models. The preference-free models are further classified as single-plus, double-plus, and triple-plus models.

Chapter 4

This chapter introduces the fundamental term structure model of Vasicek [1977] and derives the preference-free extensions of this model given as the Vasicek+ model, the Vasicek++ model (i.e., the extended Vasicek model), and the Vasicek+++ model (i.e., the fully extended Vasicek model). The chapter also obtains analytical results to price Eurodollar futures and European bond options and demonstrates the construction of binomial and trinomial trees for these models to price American bond options.

Chapter 5

This chapter extends the fundamental and preference-free Vasicek models by allowing Gaussian and exponentially distributed jumps. The chapter provides closed-form solutions for pricing zero-coupon bonds and Eurodollar futures and introduces the Fourier inversion method to price European bond options. The chapter also provides jump-diffusion trees by extending the work of Amin [1993] to price bond options with American features. These trees allow an arbitrarily large number of nodes at each step to capture the jump component, while two local nodes are used to capture the diffusion component. The chapter demonstrates how to calibrate jump-diffusion trees to fit an initial yield curve or initial zero-coupon bond prices by giving an analytical solution to the time-dependent drift of the short rate process. Finally, the chapter introduces the Fast Fourier Transform (FFT) and the fractional FFT for efficient pricing of options.

Chapter 6

This chapter begins with the derivation of formulas for bond prices, Eurodollar futures, and European bond options under the Cox, Ingersoll, and Ross

(CIR) [1985] model. Next, we show how to build binomial and trinomial trees to price derivatives with American features. As pointed out recently by Nawalkha and Beliaeva (NB) [2006], the movement of the short rate tree when the short rate hits the zero boundary has an error in the original solution proposed by Nelson and Ramaswamy (NR) [1990]. NB correct this error using a truncated-tree transform, which is then used to generate binomial and trinomial trees to price American bond options. The final section of this chapter extends the CIR model to allow jumps and demonstrates the construction of truncated jump-diffusion trees for the CIR model extended with jumps under two jump-size distributions. Under the first case, the jumps in the short rate are distributed exponentially, allowing positive jumps only; while under the second case, jumps in the short rate are distributed lognormally, allowing both positive and negative jumps.

Chapter 7

This chapter gives preference-free extensions of the fundamental CIR model and derives the solutions to bond price, Eurodollar futures, and European bond options under these extensions. The chapter considers both stationary and explosive preference-free CIR models. The chapter also introduces the constant-elasticity-of-variance (CEV) models and their preference-free extensions. Further, the chapter derives Nelson-Ramaswamy–type transforms for generating recombining trees under preference-free CEV models. Finally, the preference-free CIR and CEV models are extended with jumps.

Chapter 8

This chapter considers the maximal versions of the three subfamilies of two-factor fundamental affine models (based on the Dai and Singleton [2000] classification) and derives their preference-free extensions. The solutions of Eurodollar futures and European bond options are derived only for the $A_1(2)$ subfamily. The corresponding solutions for the $A_0(2)$ and $A_2(2)$ subfamilies are nested under the simple $A_M(N)$ affine models and are given in Chapter 9. The preference-free single-plus extension of the fundamental $A_2(2)$ model, or the $A_2(2)+$ model, is shown to allow *negative* physical correlation between the state variables, even though negative physical correlation is disallowed under the fundamental model, thus demonstrating more realistic expected return relations under the preference-free models. Finally, this chapter demonstrates how to construct two-factor trees under all three subfamilies of the two-factor affine models and their preference-free extensions. The stochastic volatility–based maximal $A_1(2)$ model uses the two-dimensional transform of Nawalkha and Beliaeva [2007] for the tree construction.

Chapter 9

The first part of this chapter considers the maximal versions of two of the four subfamilies of three-factor fundamental affine models, given as the $A_1(3)$ model and the $A_2(3)$ model. The preference-free single-plus and double-plus models are derived corresponding to the maximal fundamental models in both these subfamilies. Next, the chapter gives a general derivation of the *simple* $A_M(N)$ model and its preference-free extensions. The N-M Gaussian processes are *uncorrelated* with the M square-root processes (though the Gaussian processes may be correlated among themselves) under the simple $A_M(N)$ model. The maximal versions of the $A_0(3)$ model and the $A_3(3)$ model are subsumed in the simple $A_M(N)$ model. The simple $A_M(N)$ model allows analytical closed-form solutions of the bond price and Eurodollar futures and quasi-analytical approximations to the prices of:

1. options on zero-coupon bonds (or caplets) using the Fourier inversion method, and
2. options on coupon bonds (or swaptions) using the cumulant expansion method.

An advantage of the Fourier inversion method for pricing caplets is that it uses only a single summation to obtain solutions for models with multiple factors. Solutions under multiple-factor models typically require multiple-order summations using the traditional methods, such as Longstaff and Schwartz [1992]. The cumulant expansion method also works extremely fast for pricing swaptions, since it does not require numerical integrations. In fact, we find that the cumulant expansion method works even faster than the Fourier inversion method, and hence, it can be used to price caplets as well. However, this method slows down for instruments with a large number payments, such as options on long maturity bonds, long maturity caps, or swaptions on long maturity swaps.

The final section of this chapter gives analytical solutions for valuing credit default swaps (CDS) using simple $A_M(N)$ models. These solutions allow an arbitrary number of factors for the short rate and the default intensity and nest the solutions of Longstaff, Mithal, and Neis [2003] and Pan and Singleton [2005].

Chapter 10

This chapter introduces single-and multifactor quadratic term structure models (QTSMs) with state variables that follow Gaussian processes. The short rate under the QTSMs is expressed as a quadratic function of the state variables. Unlike the affine models, in which an N-factor affine model can belong to $N + 1$ nonnested subfamilies, an N-factor quadratic model always leads to a single maximal model that nests all other N-factor quadratic models. The chapter considers both the fundamental and the preference-free QTSMs and obtains analytical closed-form solutions to the bond price and Eurodollar futures and quasi-analytical approximations to the prices of:

1. options on zero-coupon bonds (or caplets) using the Fourier inversion method, and
2. options on coupon bonds (or swaptions) using the cumulant expansion method.

The chapter also provides basic formulas of pricing credit derivatives, such as credit default swaps, using the fundamental and the preference-free N-factor QTSMs.

Chapter 11

Unlike affine and quadratic models, which first originated as fundamental models and were subsequently generalized as preference-free models, the forward rate models of Heath, Jarrow, and Morton (HJM) [1992] and the LIBOR market model (LMM) are preference-free by construction. Since these models exogenously specify the forward rate process, which uniquely determines the risk-neutral short rate process without requiring the specification of market prices of risks, preferences do not enter in the valuation process. Further, since the initially given forward rates are taken as the model input, these models are "double-plus" by construction.

This chapter introduces the equations for the bond price, the forward rates, and the short rate under the HJM forward rate model. The chapter shows how to construct nonrecombining explosive trees and demonstrates the advantages of using the *recursive* programming technique of Das [1998] for generating these trees. The chapter also demonstrates that the non-Markovian short rate process under the proportional volatility HJM model can be transformed into a *Markovian forward price process* by using the forward measure instead of the risk-neutral measure. The Markovian forward price process allows pricing of long-maturity caps using recombining trees under the forward measure.

Chapter 12

This chapter introduces the LIBOR market model[4] (LMM) and its extensions. We derive both the lognormal forward LIBOR model (LFM) for pricing caps and the lognormal forward swap model (LSM) for pricing swaptions. The LFM and LSM provide theoretical justifications for the widely used Black formulas for pricing caps and swaptions, respectively. The LFM assumes that the discrete forward LIBOR rate follows a lognormal distribution under the numeraire associated with the given caplet maturity, while the LSM assumes that the discrete forward swap rate follows a lognormal distribution under the swap numeraire. A joint framework is also considered by deriving the LFM using a single numeraire, which leads to an approximate Black formula for pricing swaptions. Different specifications of instantaneous volatilities and correlations are considered, consistent with the double-plus and the triple-plus versions of the LFM model.

The displaced-diffusion and the CEV extensions of the LFM are shown to capture the monotonically decreasing caplet smile, while the stochastic volatility–based extensions of the LFM are shown to capture the hockey-stick shaped caplet smile. The empirical results of the stochastic-volatility-jump extension of the LFM by Jarrow, Li, and Zhao (JLZ) [2007] are discussed for pricing caps, and the JLZ model is extended to allow *time-homogeneous* risk-premiums for volatility/jump risks.

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NOTES

1. See the extended Vasicek model of Hull and White [1990] or the Vasicek++ model in Chapter 4.
2. Brigo and Mercurio [2001] summarize various double-plus models, including the CIR++ model and the G2++ model. The G2++ model was originally derived by Hull and White [1996]. The CIR++ model was originally suggested by CIR [1985, bottom paragraph, p. 395] and derived formally by Dybvig [1988, 1997] and Scott [1995]. Chapter 7 derives the CIR++ model.
3. See the *fully extended* Vasicek model of Hull and White [1990] or the Vasicek+++ model given in Chapter 4.
4. The LMM model was discovered by Brace, Gatarek, and Musiela [1997] and is also referred to as the BGM model by many practitioners. Miltersen, Sandmann, and Sondermann [1997] also discovered this model independently, and Jamshidian [1997] contributed significantly to its initial development.