Interest Rate Risk Modeling

The Fixed Income Valuation Course

Sanjay K. Nawalkha
Gloria M. Soto
Natalia A. Beliaeva

- **Interest Rate Risk Modeling : The Fixed Income Valuation Course.** Sanjay K. Nawalkha, Gloria M. Soto, Natalia K. Beliaeva, 2005, Wiley Finance.
  - **Chapter 10 :**

    Principal Component Model with VaR Analysis

- **Goals:**
  - Learn the techniques of principal component analysis
  - Use PC model to implement VaR analysis
  - Understand the limitations of the PC model
  - Know how to apply PCA to mortgage-backed securities or other securities

Chapter 10 :
Principal Component Model with VaR Analysis

- Introduction
- From Term Structure Movements to Principal Components
- Principal Component Durations and Convexities
- Risk Measurement and Management with the Principal Component Model
- VaR Analysis Using the Principal Component Model
- Limitations of the Principal Component Model
- Applications to Mortgage Securities

Chapter 10 :
Principal Component Model with VaR Analysis

- Introduction
- From Term Structure Movements to Principal Components
- Principal Component Durations and Convexities
- Risk Measurement and Management with the Principal Component Model
- VaR Analysis Using the Principal Component Model
- Limitations of the Principal Component Model
- Applications to Mortgage Securities

Introduction

- There are two reasons that in previous chapters we derived interest rate risk hedging conditions without explicitly modeling the historical information contained in the factor structure of the interest rate changes:
  - The first reason: The specific term structure shifts corresponding to the first three duration vector elements capture much of the variance.
  - The second reason: The duration vector model and the key rate duration model don’t require a stationary factor structure for interest rate changes.

Introduction

- However, there exist conditions under which a principal component may be preferred to the duration vector model or the key rate duration model:
  - First, if the covariance structure of interest rates remains stationary.
  - Second, a gain in hedging efficiency.
  - Last, it allows short positions, which minimizes the immunization risk with the knowledge of the factor structure of interest rate changes.

Introduction

- The principal component model assumes that the term structure movements can be summarized by a few composite variables.

- These new variables are constructed by applying a statistical technique called **principal component analysis (PCA)** to the past interest rate changes.

- The use of PCA in the Treasury bond markets has revealed three principal components are sufficient in explaining the variation in the interest rate changes.

Introduction

- The benefits of using the PCA are:
  - A significant **reduction in dimensionality** when compared with other models.
  - It is able to produce **orthogonal risk factors**, this feature makes interest rate risk measurement and management a simpler task, because each risk factor can be treated independently.

Chapter 10 :
Principal Component Model with VaR Analysis

- Introduction
- From Term Structure Movements to Principal Components
- Principal Component Durations and Convexities
- Risk Measurement and Management with the Principal Component Model
- VaR Analysis Using the Principal Component Model
- Limitations of the Principal Component Model
- Applications to Mortgage Securities

From Term Structure Movements to Principal Components

- In the previous chapter, we modeled the term structure shift as a function of a vector of key rate changes:

$$TSIR\ shift = \left(\Delta y(t_1), \Delta y(t_2) .., \Delta y(t_m)\right) \qquad (10.1)$$

- The PCA approach provides an alternative representation of TSIR shifts by using principal components:

$$TSIR\ shift = \left(\Delta c_1, \Delta c_2 ..., \Delta c_m\right) \qquad (10.2)$$

From Term Structure Movements to Principal Components

- The principal components are linear combination of interest rate changes:

$$\Delta c_j = \sum_{i=1}^{m} u_{ji} \Delta y(t_i) \quad j = 1, \ldots, m \qquad (10.3)$$

where u_{ji} are called principal component coefficient.

From Term Structure Movements to Principal Components

- The components have unequal significance:
  - The first principal component explains the maximum percentage of the total variance of interest rate changes.
  - The second component is linearly independent of the first, and explains the maximum percentage of the remaining variance
  - And so on…

From Term Structure Movements to Principal Components

- The principal components are constructed using the covariance matrix of zero-coupon rate changes.

- Since this matrix is symmetric by construction, it must have m normalizes and linearly independent eigenvectors, $U_1, \ldots, U_m$, corresponding to m positive eigenvalues, $\lambda_1 \ldots, \lambda_{m.}$

- The proportion of the variance explained by the jth principal component is:

$$\frac{\lambda_j}{\sum_{j=1}^{m} \lambda_j} \qquad (10.5)$$

FixedIncomeRisk.com

From Term Structure Movements to Principal Components

- Using the equation 10.3, the changes in the m interest rates can be obtained as follows:

$$\Delta y(t_i) = \sum_{j=1}^{m} u_{ji} \Delta c_j \qquad i = 1, \ldots, m \qquad (10.6)$$

- The principal components with low eigenvalues make little contribution in explaining, hence these components can be removed without losing significant information.

- This not only helps obtaining a **low-dimensional parsimonious model**, but also reduces the noise in the data due to unsystematic factors.

From Term Structure Movements to Principal Components

- Assuming that we retain the first k components, expression 10.6 can be written as:

$$\Delta y(t_i) = \sum_{j=1}^{k} u_{ji} \Delta c_j + \varepsilon_i \qquad i = 1, \ldots, m \qquad (10.7)$$

where ε_i is an error term that measures the changes not explained by the k principal components.

From Term Structure Movements to Principal Components

- Table 10.1 shows the eigenvectors and eigenvalues of the covariance matrix of monthly changes in the U.S. zero-coupon rates from Jan. 2000 to Dec. 2002.

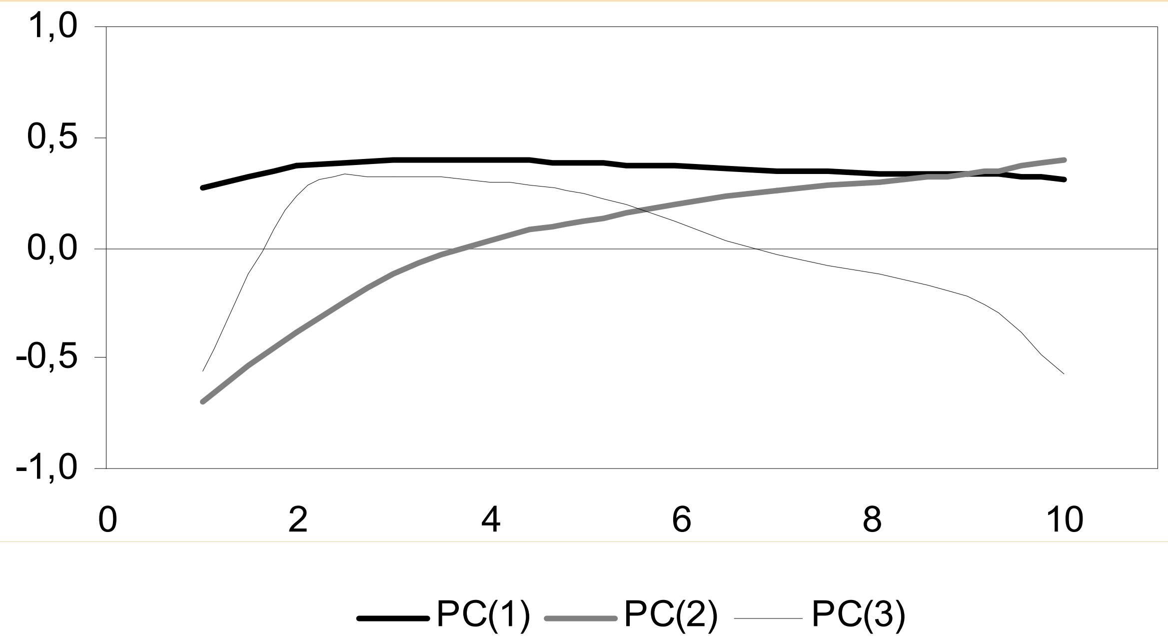
Rates	PC(1)	PC(2)	PC(3)	PC(4)	PC(5)	PC(6)	PC(7)	PC(8)
1	0.270	-0.701	-0.565	0.292	-0.138	-0.085	0.060	-0.026
2	0.372	-0.385	0.227	-0.423	0.459	0.445	-0.240	0.132
3	0.396	-0.120	0.315	-0.328	-0.037	-0.605	0.244	-0.442
4	0.395	0.028	0.296	0.103	-0.411	-0.182	-0.054	0.735
5	0.382	0.124	0.243	0.346	-0.415	0.476	-0.166	-0.483
7	0.350	0.252	-0.031	0.344	0.444	0.149	0.682	0.102
9	0.332	0.334	-0.225	0.266	0.397	-0.348	-0.614	-0.047
10	0.312	0.393	-0.576	-0.556	-0.270	0.162	0.085	0.022
Eigenvalues	0.605	0.057	0.009	0.001	0.001	0.000	0.000	0.000

From Term Structure Movements to Principal Components

- Using Equation 10.5, we can calculate the proportion of this variance explained by the jth principal component:
  - The first three principal components explain almost all of the variance of interest rate changes, a result consistent with other studies.
  - The first factor accounts for 89.8% of the total variance, while the second and third factors account for 8.5% and 1.3%.
  - In sum, the first three components explain 99.6% of the variability of the data.

From Term Structure Movements to Principal Components

- Figure 10.1 shows the shape of the eigenvectors corresponding to the first three components:
  - The change in the zero-coupon rates (on y-axis) is plotted against the maturity terms (on x-axis) with respect to each principal component.



From Term Structure Movements to Principal Components

- A better approach to modify the model to make each factor have a unit variance:
  - This is achieved by multiplying each eigenvector by the square root of its eigenvalue, and dividing the principal component by the square root of the eigenvalue.

$$\Delta y(t_i) = \sum_{j=1}^{k} \left(u_{ji} \sqrt{\lambda_j} \right) \Delta c_j^{*} + \varepsilon_i \qquad i = 1, \ldots, m \quad (10.8)$$

$$\Delta c_j^{*} = \frac{\Delta c_j}{\sqrt{\lambda_j}}$$

From Term Structure Movements to Principal Components

- The coefficient in parenthesis, which measure the impact of a one standard deviation move in each principal component on each interest rate, are called **factor loadings**.
- A simpler notation, and using only three factors can be approximated as follows:

$$\Delta y(t_i) \approx l_{ih}\Delta c_h + l_{is}\Delta c_s + l_{ic}\Delta c_c \qquad i = 1,\ldots,m \qquad (10.9)$$

$$\Delta c_h = \Delta c_1^* = \frac{\Delta c_1}{\sqrt{\lambda_1}}, \qquad \Delta c_s = \Delta c_2^* = \frac{\Delta c_2}{\sqrt{\lambda_2}} \qquad \Delta c_c = \Delta c_3^* = \frac{\Delta c_3}{\sqrt{\lambda_3}}$$

$$l_{ih} = u_{1i}\sqrt{\lambda_1}, \qquad l_{is} = u_{2i}\sqrt{\lambda_2}, \qquad l_{ic} = u_{3i}\sqrt{\lambda_3}$$

Chapter 10 :
Principal Component Model with VaR Analysis

- Introduction
- From Term Structure Movements to Principal Components
- Principal Component Durations and Convexities
- Risk Measurement and Management with the Principal Component Model
- VaR Analysis Using the Principal Component Model
- Limitations of the Principal Component Model
- Applications to Mortgage Securities

Principal Component Durations and Convexities

- Principal component durations and convexities can be computed from the first and the second partial derivatives of the security with respect to the three factors as follows:

$$PCD(\ ) = -\frac{1}{P}\frac{\partial P}{\partial c_i} \qquad i = h,s,c \qquad (10.10)$$

$$PCC(\,, j) = \frac{1}{P}\frac{\partial^2 P}{\partial c_i \partial c_j} \qquad i,j = h,s,c \qquad (10.11)$$

Principal Component Durations and Convexities

- Using the second-order Taylor series approximation:

$$\frac{\Delta P}{P} = -\sum_{i=h,s,c} PCD(i) \cdot \Delta c_i + \frac{1}{2} \sum_{i=h,s,c} \sum_{j=h,s,c} PCC(i,j) \cdot \Delta c_i \cdot \Delta c_j \quad (10.12)$$

- Since the principal components are independent, we can simplify by disregarding the cross effects, which gives:

$$\frac{\Delta P}{P} = -\sum_{i=h,s,c} PCD(i) \cdot \Delta c_i + \frac{1}{2} \sum_{j=h,s,c} PCC(i,i) \cdot \Delta c_i^2 \quad (10.13)$$

Principal Component Durations and Convexities

- The principal component measures can also be computed directly from key rate durations and convexities, disregarding cross effects of principal component shifts:

$$\frac{\Delta P}{P} = -\sum_{v=h,s,c} \Delta c_v \sum_{i=1}^{m} KRD(i) \cdot l_{iv}$$

$$+\frac{1}{2} \sum_{v=h,s,c} \Delta c_v^2 \sum_{i=1}^{m} \sum_{j=1}^{m} KRC(i,j) \cdot l_{iv} \cdot l_{jv} \qquad (10.14)$$

Principal Component Durations and Convexities

- Comparing equation 10.13 with 10.14, the principal component durations (convexities) can be expressed as linear combinations of the key rate durations (convexities) :

$$PCD(v) = \sum_{i=1}^{m} KRD(i)\, l_{iv}$$

$$PCC(v) = \sum_{i=1}^{m} \sum_{j=1}^{m} KRC(i,j)\, l_{iv} \cdot l_{jv} \qquad v = h,s,c \qquad (10.15)$$

Principal Component Durations and Convexities

- Though the traditional duration equals the sum of the key rate durations, the height-factor PCD may not always coincide with traditional duration. Because…
  - The first principal component does not provide an exact parallel TSIR movement.
  - Even with a parallel move, normalizing the eigenvectors prevents this equivalence unless adjusted by a proportionality factor.

Principal Component Durations and Convexities

- **Example 10.1**

  Reconsider the five bonds in Chapter 9, whose main characteristics and key rate durations with respect to the 1,2,3,4, and five-year zero-coupon rates are reproduced in Table 10.2. as follows:

	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5
Face value	$1,000	$1,000	$1,000	$1,000	$1,000
Maturity (years)	1	2	3	4	5
Annual coupon rate (%)	10	10	10	10	10
Price	$1,046.35	$1,080.54	$1,110.42	$1,137.62	$1,162.74
KRD(1)	1	0.088	0.086	0.084	0.082
KRD(2)	0	1.824	0.161	0.157	0.154
KRD(3)	0	0	2.501	0.222	0.217
KRD(4)	0	0	0	3.055	0.272
KRD(5)	0	0	0	0	3.504

Principal Component Durations and Convexities

- The level of the key rates is assumed to be 5%, 5.5%, 5.75%, 5.9% and 6%.
- Factor loadings shown in Table 10.3 as follows, are obtained from Table 10.1 by multiplying the eigenvector of each principal component by the squared root of the corresponding eigenvalue.

Years	PC(1)	PC(2)	PC(3)
1	0.210	-0.168	-0.054
2	0.289	-0.092	0.022
3	0.308	-0.029	0.030
4	0.307	0.007	0.028
5	0.297	0.030	0.023

Principal Component Durations and Convexities

- Principal component durations for each bond are computed using equation 10.15 and are shown in Table 10.4 together with traditional duration.

$$PCD(v) = \sum_{i=1}^{m} KRD(i)\, l_{iv}$$

$$PCC(v) = \sum_{i=1}^{m}\sum_{j=1}^{m} KRC(i,j)\, l_{iv} \cdot l_{jv} \qquad v = h,s,c \qquad (10.15)$$

	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5
PCD(*h*)	0.210	0.546	0.834	1.070	1.254
PCD(*s*)	-0.168	-0.183	-0.101	-0.014	0.071
PCD(*c*)	-0.054	0.035	0.074	0.091	0.094
D	1.000	1.912	2.748	3.518	4.229

Principal Component Durations and Convexities

- In Table 10.4, we can see that the height-factor duration, PCD(h), increase with the bond's maturity since the exposure to a near parallel component of the term structure shift must increase with maturity.

- The **scaled principal component durations** are 0.765, 1.985, 3.033, 3.891, and 4.560, which are similar to the traditional duration of the bonds given in the last row of Table 10.4.

Principal Component Durations and Convexities

- Principal component risk measures can be obtained similarly form the key rate measures of the portfolio or as a weighted average of the principal component measures of each security in the portfolio.

- To illustrate, reconsider the **ladder, barbell, and bullet** bond portfolios in Chapter 9, whose key rate duration vectors and portfolio's composition are reproduced in Table 10.5.
  - A ladder has similar key rate durations across the maturity range
  - A barbell (bullet) has high (low) key rate durations to the short and long interest rates, and low (high) durations for the intermediate rates.

Principal Component Durations and Convexities

- Table 10.5

	Ladder	Barbell	Bullet
Bond 1	0.2	0.479	0.000
Bond 2	0.2	0.000	0.521
Bond 3	0.2	0.000	0.000
Bond 4	0.2	0.000	0.479
Bond 5	0.2	0.521	0.000
KRD(1)	0.268	0.522	0.086
KRD(2)	0.459	0.080	1.025
KRD(3)	0.588	0.113	0.106
KRD(4)	0.665	0.141	1.464
KRD(5)	0.701	1.825	0.000

Principal Component Durations and Convexities

- As mentioned earlier, the portfolio's key rate duration vectors together with the factor loadings in Table 10.3 allow the component durations displayed in Table 10.6.

	Ladder	**Barbell**	**Bullet**
PCD(*h*)	0.783	0.754	0.797
PCD(*s*)	-0.079	-0.043	-0.102
PCD(*c*)	0.048	0.023	0.062

- According to the absolute values of the measures, the bullet portfolio shows the highest risk exposure to the level, slope, and curvature shifts in the term structure, while the barbell portfolio show the lowest risk exposure.

Chapter 10 :
Principal Component Model with VaR Analysis

- Introduction
- From Term Structure Movements to Principal Components
- Principal Component Durations and Convexities
- Risk Measurement and Management with the Principal Component Model
- VaR Analysis Using the Principal Component Model
- Limitations of the Principal Component Model
- Applications to Mortgage Securities

Risk Measurement and Management with the Principal Component Model

- Key rate durations are a valid starting point for avoiding the ambiguity inherent in principal component shifts.

- Managers can benefit from the intuitive description of risk provided by the key rate durations.

- Managers can also benefit from the principal component durations for implementing more parsimonious portfolio strategies that do not exhaust all degrees of freedom in portfolio construction.

Risk Measurement and Management with the Principal Component Model

- Immunizing a portfolio for a given horizon requires choosing a portfolio composition where the three durations of the portfolio equals a zero-coupon bond's duration maturing at the end of the planning horizon:

$$PCD(i)_{PORT} = PCD(i)_{zero} = H \cdot l_{Hi} \quad i = h,s,c \qquad (10.17)$$

where H is the length of the planning horizon and l_{Hi} is the loading of principal component I on continuously compounded zero-coupon rate for term H.

Risk Measurement and Management with the Principal Component Model

- **Example 10.2**

  Chapter 9 Example 9.3 demonstrated how a portfolio could be immunized with five key rates using six different bonds.
- The same six-bond portfolio can be immunized using the principal component model.
  - In this case, the immunization constraints are given as follows:

$$PCD(h) = p_1 \cdot PCD_1(h) + p_2 \cdot PCD_2(h) + \ldots + p_6 \cdot PCD_6(h) = H \cdot I_{Hh}$$

$$PCD(s) = p_1 \cdot PCD_1(s) + p_2 \cdot PCD_2(s) + \ldots + p_6 \cdot PCD_6(s) = H \cdot I_{Hs}$$

$$PCD(c) = p_1 \cdot PCD_1(c) + p_2 \cdot PCD_2(c) + \ldots + p_6 \cdot PCD_6(c) = H \cdot I_{Hc}$$

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$$

Risk Measurement and Management with the Principal Component Model

- The number of the bonds (6) exceeds the number of constraints (3), the system of equations has an infinite number of solutions for the bond proportions p_1, p_2, ... p_6

- To select a unique immunizing solution, we optimize by using the quadratic function:

$$\text{Min}\left[\sum_{i=1}^{6} p_i^{\,2}\right]$$

Risk Measurement and Management with the Principal Component Model

- Expressing these conditions in matrix form and some matrix manipulation gives the following solution:

$$
\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_6 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}
=
\begin{bmatrix}
2 & 0 & \cdots & 0 & PCD_1(h) & PCD_1(s) & PCD_1(c) & 1 \\
0 & 2 & \cdots & 0 & PCD_2(h) & PCD_2(s) & PCD_2(c) & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 2 & PCD_6(h) & PCD_6(s) & PCD_6(c) & 1 \\
PCD_1(h) & PCD_2(h) & \cdots & PCD_6(h) & 0 & 0 & 0 & 0 \\
PCD_1(s) & PCD_2(s) & \cdots & PCD_6(s) & 0 & 0 & 0 & 0 \\
PCD_1(c) & PCD_2(c) & \cdots & PCD_6(c) & 0 & 0 & 0 & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0
\end{bmatrix}^{-1}
\cdot
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ H \cdot I_{Hh} \\ H \cdot I_{Hs} \\ H \cdot I_{Hc} \\ 1 \end{bmatrix}
$$

Risk Measurement and Management with the Principal Component Model

- The first six elements of the left column vector give the proportions to be invested in the six bonds. We obtained the following solution for the bond proportions:

$$p_1 = -0.099 \quad p_2 = -0.215 \quad p_3 = 0.525$$

$$p_4 = 0.498 \quad p_5 = 0.158 \quad p_6 = 0.133$$

- The principal component solution is more diversified than the immunized portfolio obtained under the key rate model.

Risk Measurement and Management with the Principal Component Model

- Immunizing using a **key rate model** resembles dedication strategies leading to near perfect hedging performance, with zero net cash flows.

- While the **principal component model**'s main strength is its low dimension leading to lower transaction costs and higher degrees of freedom.

Chapter 10 :
Principal Component Model with VaR Analysis

- Introduction
- From Term Structure Movements to Principal Components
- Principal Component Durations and Convexities
- Risk Measurement and Management with the Principal Component Model
- VaR Analysis Using the Principal Component Model
- Limitations of the Principal Component Model
- Applications to Mortgage Securities

VaR Analysis Using the Principal Component Model

- VaR analysis using the principal component model has some advantages over the key rate model:
  - The principal components are uncorrelated by construction
  - The correlation matrix of principal components is the identity matrix
  - Principal components are normally distributed

VaR Analysis Using the Principal Component Model

- The approximation of portfolio returns based on principal component durations is then normally distributed with variance equal to:

$$\sigma^2_R = \sum_{j=h,s,c} PCD(i)^2 \qquad (10.18)$$

- Using equation 10.18 and the definition of VaR, we obtain the VaR of a portfolio at a c percentage confidence as:

$$VaR_c = V_0 z_c \sqrt{\sum_{i=h,s,c} PCD(i)^2} \qquad (10.19)$$

where V_0 is the initial market value of the portfolio and z_c is the *c* percentile of a standard normal distribution.

VaR Analysis Using the Principal Component Model

- **Example 10.3**

  We are going to construct the VaR measurement using the principal component model based on the same data in Chapter 9.

  The VaR values at the 95% and 99% levels for each portfolio using the principal component model are given as follows:

$$VaR_{95} = 10{,}000 \times 1.645 \times \sqrt{PCD_{PORT}(h)^2 + PCD_{PORT}(s)^2 + PCD_{PORT}(c)^2}$$

$$VaR_{99} = 10{,}000 \times 2.326 \times \sqrt{PCD_{PORT}(h)^2 + PCD_{PORT}(s)^2 + PCD_{PORT}(c)^2}$$

VaR Analysis Using the Principal Component Model

- Table 10.7 shows the monthly standard deviation of the portfolio returns and the VaR numbers:

	Ladder	Barbell	Bullet
σ_R	0.788	0.755	0.806
VaR_{95}	\$129.67	\$124.26	\$132.56
VaR_{99}	\$183.40	\$175.74	\$187.48

- The result is consistent as in Example 10.1. The bullet portfolio is the riskiest, followed by the ladder and finally the barbell portfolio.

VaR Analysis Using the Principal Component Model

- Table 10.7 differ only slightly from the key rate model. This demonstrates the principal component model is able to provide an accurate description of interest rate dynamics while maintaining a low number of risk factors.

Chapter 10 :
Principal Component Model with VaR Analysis

- Introduction
- From Term Structure Movements to Principal Components
- Principal Component Durations and Convexities
- Risk Measurement and Management with the Principal Component Model
- VaR Analysis Using the Principal Component Model
- Limitations of the Principal Component Model
- Applications to Mortgage Securities

Limitations of the Principal Component Model

- **Limitations of the Principal Component Model**
  - Static Factors Arising from a Dynamic Volatility Structure
  - Principal component Analysis: Using Zero-Coupon Rate Changes or Forward Rate Changes

Limitations of the Principal Component Model

- The principal component model has a couple of shortcomings given as follows:
  - First, the static nature of the technique is unable to deal with the non-stationary time-series behavior of the interest rate changes.
  - Second, principal components are purely constructions that summarize information in correlated systems, but do not always lead to an economic interpretation.

Limitations of the Principal Component Model

Static Factors Arising from a Dynamic Volatility Structure

- Application of PCA to term structure movements implies that:
  - The covariance structure of interest rate changes is constant
  - The shape of principal components are stationary
- These are critical, because if the shapes of the principal components change frequently, then these components cannot explain the future volatility of interest rate changes.

Limitations of the Principal Component Model

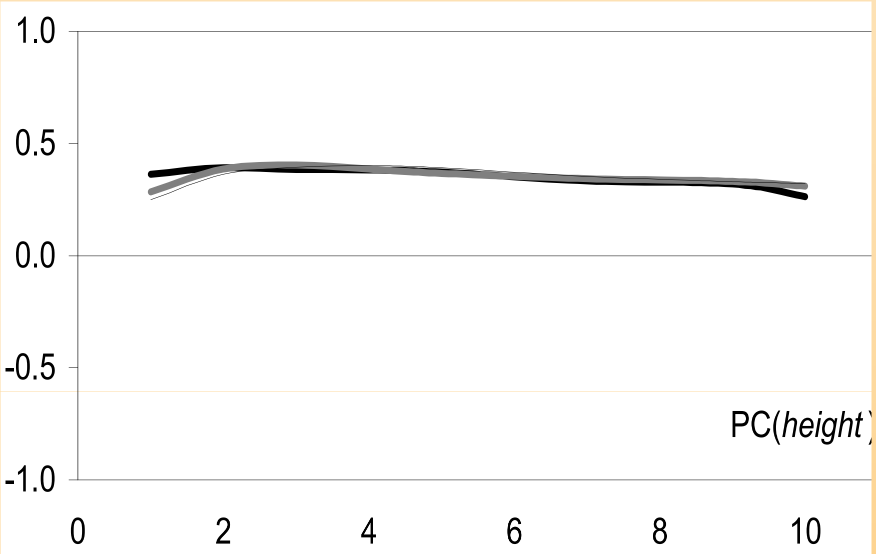
Static Factors Arising from a Dynamic Volatility Structure

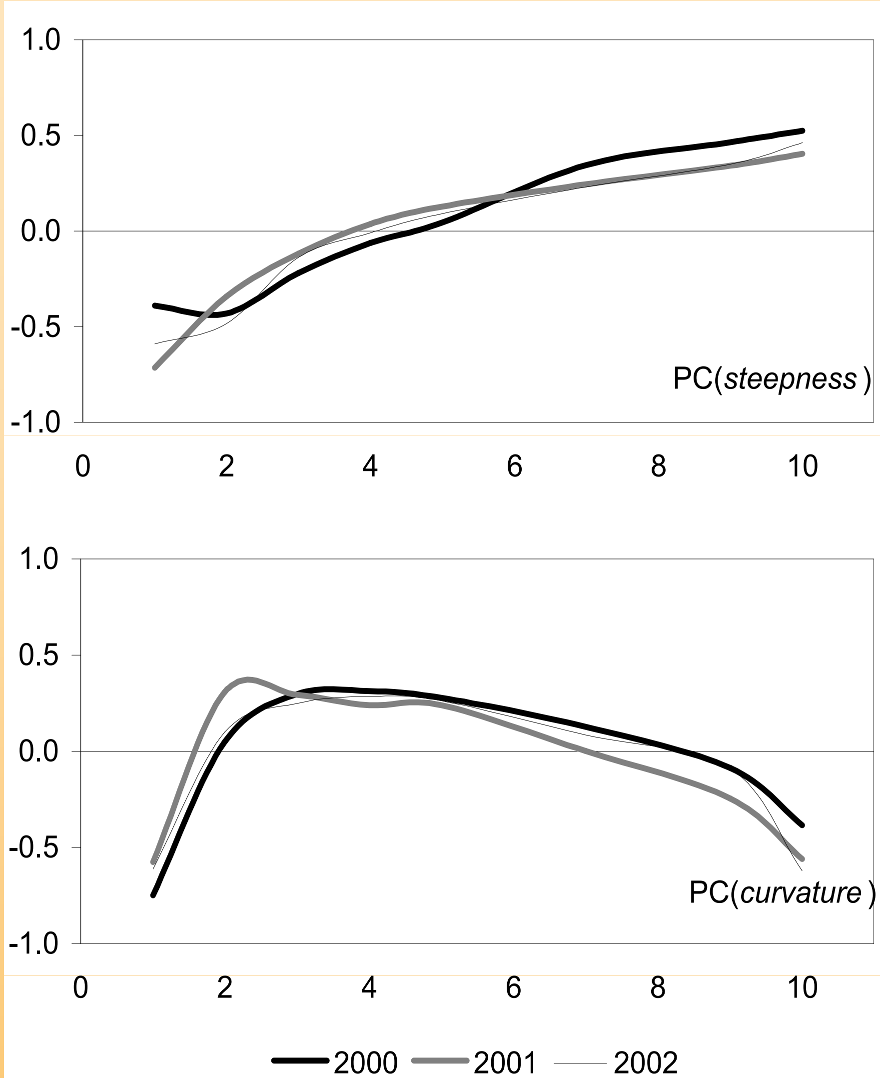
- According to Bliss(1997a) or Soto (2004b), the dynamic pattern in the volatility of interest rates affects the stability of the principal components.

- Figure 10.2 illustrates the effect of the changing volatilities of U.S. zero-coupon rates on the principal components obtained for 2000, 2001, 2002.

Limitations of the Principal Component Model

Static Factors Arising from a Dynamic Volatility Structure





Limitations of the Principal Component Model

- **Limitations of the Principal Component Model**
  - Static Factors Arising from a Dynamic Volatility Structure
  - Principal component Analysis: Using Zero-Coupon Rate Changes or Forward Rate Changes

Limitations of the Principal Component Model

Using Zero-Coupon Rate Changes or Forward Rate Changes

- Lekkos (2000) attributes the shape of threes three components to the aggregating process of computing zero-coupon rates from forward rates.
- Assume that all forward rate changes have a unit variance, and zero correlations with each other.
- Application of PCA on the covariance matrix including the 1 to 10-year zero-coupon rates reveals that the first principal component accounts for 69.7% of the total variance, first two account for 87.5%,and the first three account for 93.9%.

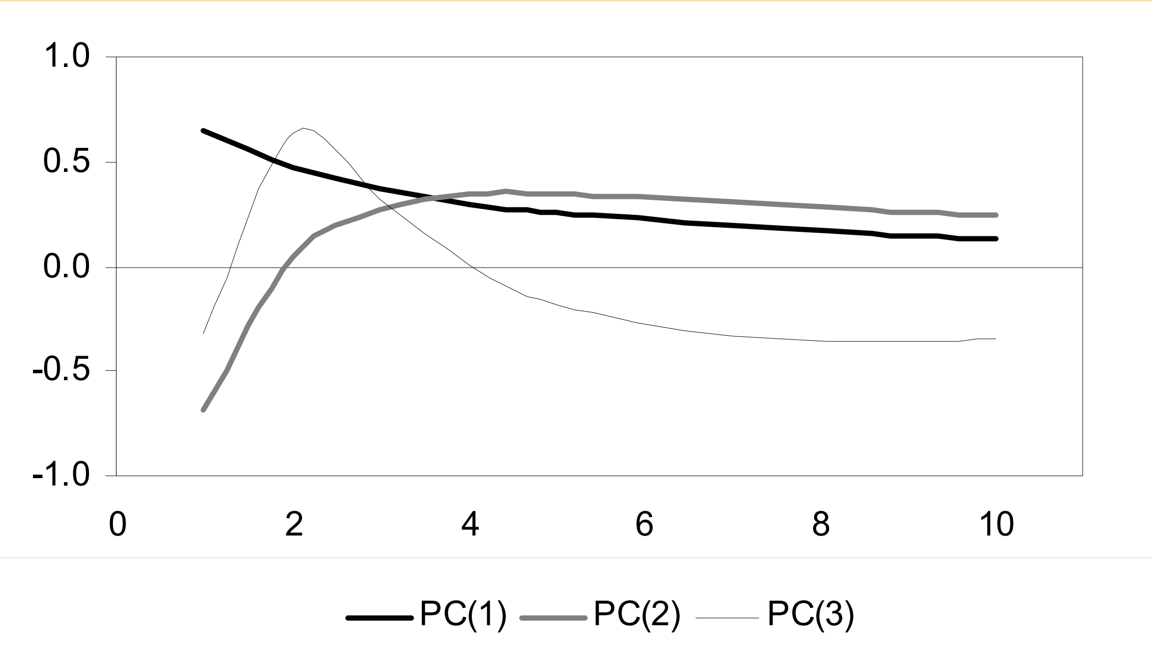
Limitations of the Principal Component Model
Using Zero-Coupon Rate Changes or Forward Rate Changes

- Only the first three principal components are needed to explain almost the entire variance of zero-coupon rate changes even though all forward rate changes have zero correlation by definition.
- The shape of these three principal components is shown in Figure 10.3.

Limitations of the Principal Component Model

Using Zero-Coupon Rate Changes or Forward Rate Changes

- The traditional interpretation of the three principal components as **height**, **slope**, and **curvature factors** could simply be due to the definition of **zero-coupon rates as aggregates of forward rates**.



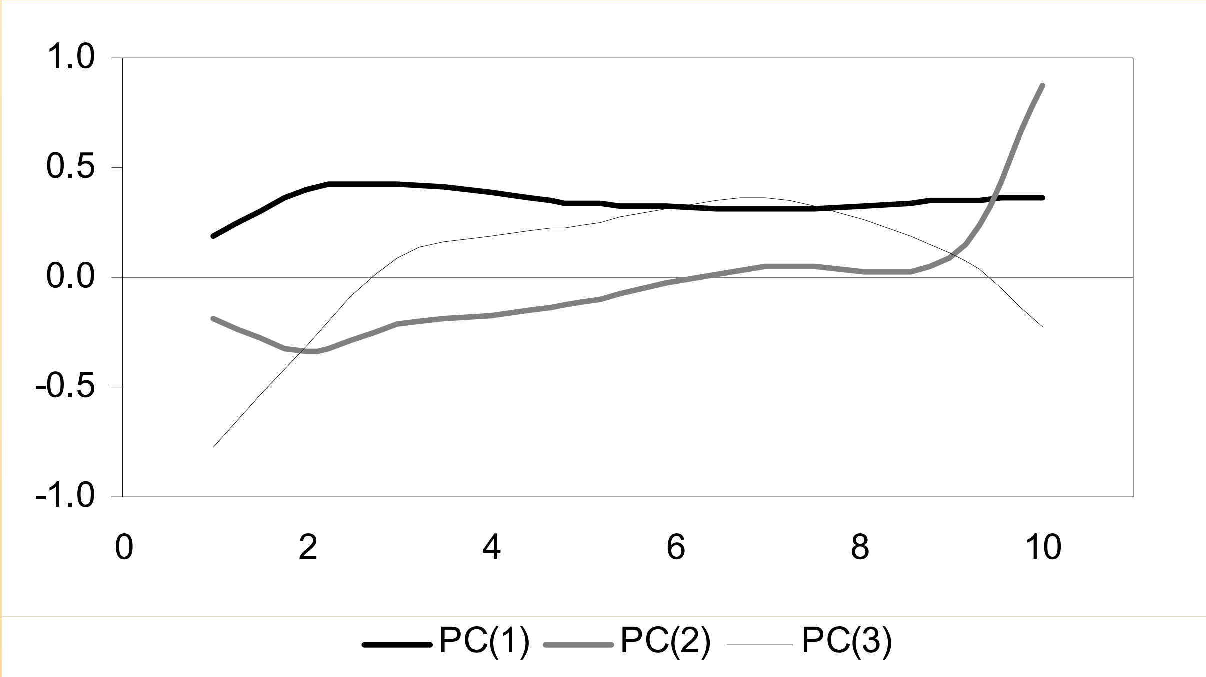
Limitations of the Principal Component Model

Using Zero-Coupon Rate Changes or Forward Rate Changes

- More importantly, in this aggregation process some information about the variability in the interest rates may be lost.
- Using principal component analysis on forward rate changes based upon real market data, **a minimum six factors** are needed to account for **99%** of the total variability of rate changes.
- The first three principal components of forward rate changes for the period Jan. 2000 to Dec. 2002 are shown in Figure 10.4.

Limitations of the Principal Component Model

Using Zero-Coupon Rate Changes or Forward Rate Changes



- The variability is spread out more evenly across the first six principal components using the forward rate changes, and the first principal component explains only 59% of the variability.

Chapter 10 :
Principal Component Model with VaR Analysis

- Introduction
- From Term Structure Movements to Principal Components
- Principal Component Durations and Convexities
- Risk Measurement and Management with the Principal Component Model
- VaR Analysis Using the Principal Component Model
- Limitations of the Principal Component Model
- Applications to Mortgage Securities

Applications to Mortgage Securities

- In this section we show how to apply the PCA model to compute the empirical PC durations of **mortgage-backed securities (MBS**), and we focus on pass-through securities.

- The U.S. mortgage securities market is the largest debt market in the world.

- The **three main products** under MBS:
  - Mortgage pass-through securities
  - Collateralized mortgage obligations (CMOs)
  - Stripped MBS, interest only(IO)/principal only (PO)

Applications to Mortgage Securities

- MBS have high degree of optionality, which makes them difficult to value and hedge.

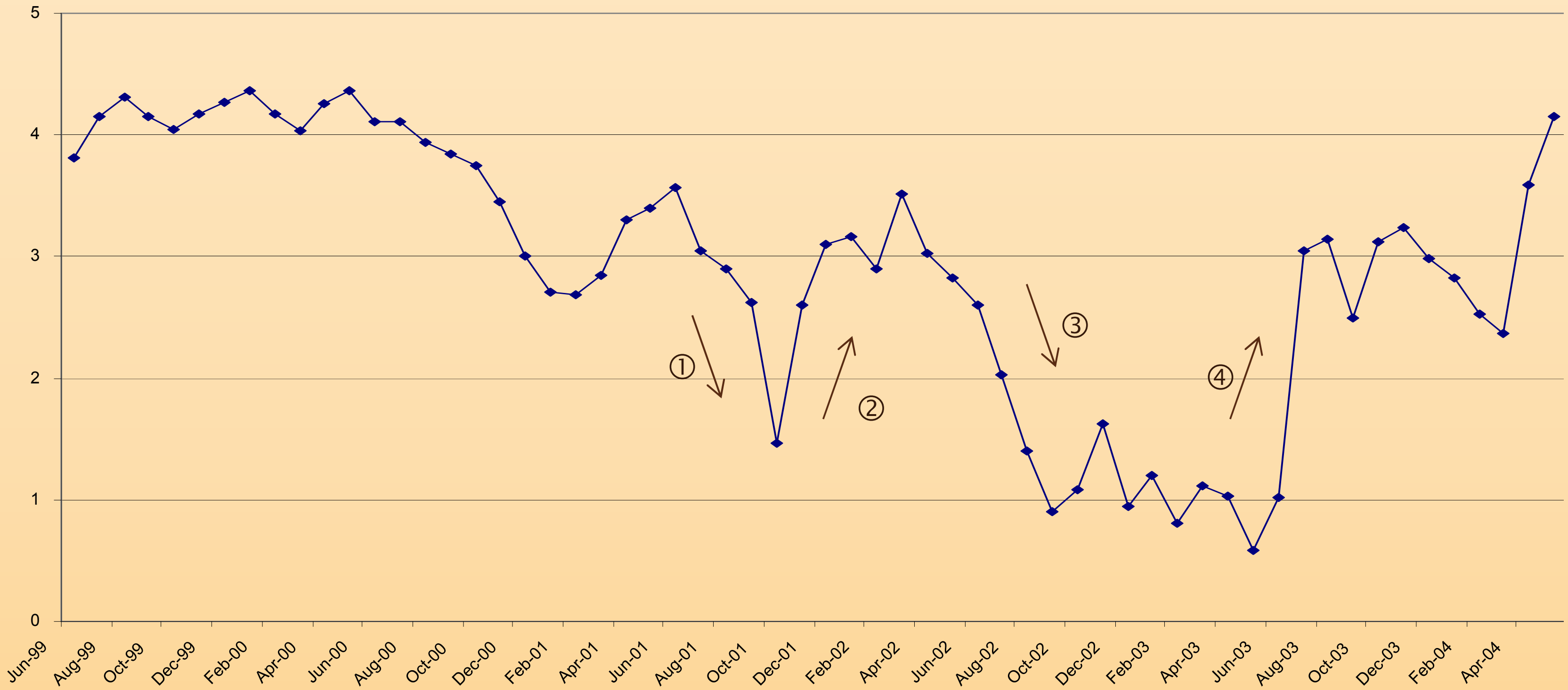
- The **prepayment option** is generally a lot more significant in the valuation and hedging of MBS than the default option:
  – **Interest rates** ↓: most homeowners prepay through refinancing, which shortens the duration of the mortgage securities.
  – **Interest rates** ↑: the likelihood of prepayments reduces, as homeowners hold on to their mortgage loans financed at low rates, lengthening the duration of the mortgage securities.

Applications to Mortgage Securities

- Due to the prepayment option MBS have tremendous **negative convexity**:
  - **Interest rates ↓**: the MBS prices increase, but prepayments put an upper limit on the upside movements in the prices
  - **Interest rates ↑**: the probability of prepayment becomes lower, and hence prices fall more rapidly with lengthening duration

- Figure 10.5 shows the monthly changes in duration for the Lehman Brothers MBS Index from June 1999 to June 2004.

Applications to Mortgage Securities



① Over 2001: 11 interest rate cuts by FOMC

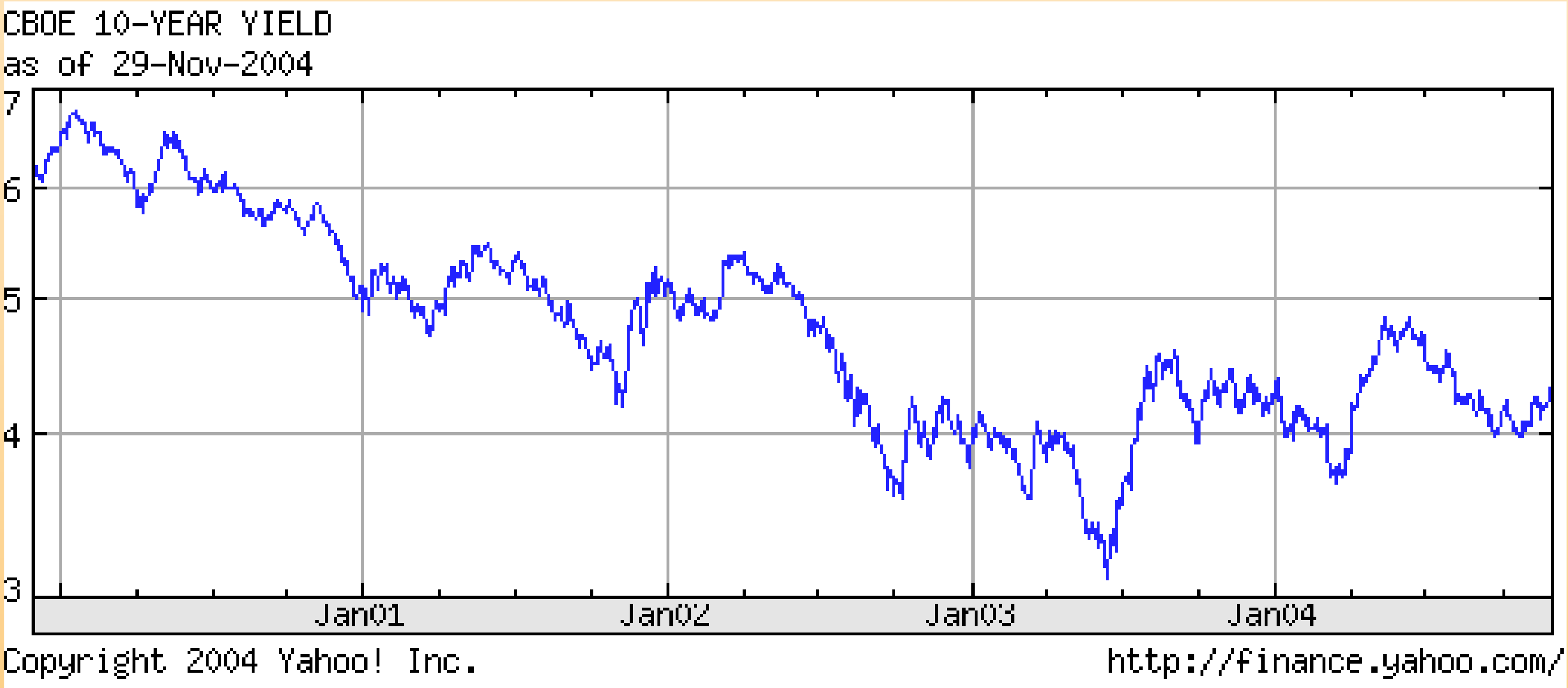
② Beginning 2002: expectation of an economic rebound

③ Late 2002: Iraq war

④ 2003: end of the Iraq war

Applications to Mortgage Securities

- Figure 10.6 Ten-Year U.S. T-Bonds Yield



Applications to Mortgage Securities

- The tremendous variability in the duration of the Lehman MBS index is caused by the significant influence of the **prepayment option** on the interest rate sensitivities of MBS.

- Capturing the nonstationarity in the duration of MBS is crucial in managing the interest rate risk of a mortgage portfolio.

- In the following we are going to implement an empirical PC duration.

Applications to Mortgage Securities

- The empirical estimation of PC durations is done in two stages:
  - **First stage**:

    Compute the principal component factors using the methodology outlined earlier in this chapter with six months of daily rate changes in the key U.S. Treasury rates
  - **Second stage**:

    Obtain the empirical PC durations of MBS by running time-series linear regressions of daily returns on the changes in the PC factors obtained in the first stage

Applications to Mortgage Securities

- **Applications to Mortgage Securities**
  - First Stage: Estimation of Principal Components
  - Second Stage: Estimation of Empirical PC Durations

Applications to Mortgage Securities

First Stage: Estimation of Principal Components

- The sample includes Treasury rate changes from year 1997 to year 2004. However, at any given point of time, only six months of past daily data is used to estimate the principal components.
- Table 10.8 shows the first few days of daily rate changes.
- Table 10.9 shows the covariance matrix of daily rate changes.

Applications to Mortgage Securities

First Stage: Estimation of Principal Components

- Table 10.8 the first few days of daily rate changes beginning from Jul. 18th 2003, until Jan. 12th 2004 (contains 120 business days).

Table 10.8 Changes in daily interest rates

	Change in Rate (%)				
	3-month rate	*1-year rate*	*2-year rate*	*5-year rate*	*10-year rate*
07/18/03	0.005	0.024	0.026	0.028	0.02
07/21/03	0.016	0.038	0.107	0.188	0.195
07/22/03	0.015	-0.034	-0.04	-0.022	-0.02
07/23/03	-0.01	-0.007	-0.049	-0.053	-0.048
…	…	…	…	…	…
2001/12/4	0	0.015	0	-0.01	-0.004

Applications to Mortgage Securities

First Stage: Estimation of Principal Components

- Table 10.9 the covariance matrix of daily rate changes over the same period in Table10.8.

	3m	*1y*	*2y*	*5y*	*10y*
3m	0.00016	0.00007	0.00013	0.00022	0.00023
1y	0.00007	0.00208	0.00288	0.00319	0.00277
2y	0.00013	0.00288	0.00528	0.00577	0.00499
5y	0.00022	0.00319	0.00577	0.00723	0.0063
10y	0.00023	0.00277	0.00499	0.0063	0.00595

Applications to Mortgage Securities

First Stage: Estimation of Principal Components

- The loadings are obtained in Table 10.10, the matrix shows how the PCs are obtained as linear weighted averages of daily changes in rates of different maturities.
- For example, the first PC is obtained by giving 0.018 weight to the change in the 3-month rate.

	3m	*1y*	*2y*	*5y*	*10y*
PC 1	0.018	0.286	0.506	0.607	0.541
PC 2	0.101	-0.619	-0.51	0.234	0.54
PC 3	0.072	0.721	-0.575	-0.166	0.342
PC 4	-0.056	-0.123	0.375	-0.738	0.544
PC 5	0.991	-0.001	0.106	-0.065	-0.059

Applications to Mortgage Securities

First Stage: Estimation of Principal Components

- Since we have only 5 key rates, the 5 principal components explain 100% of the variance.
- Most of the important information of rate changes is captured by the first two or three components, so the other components can be ignored.
- Over the whole sample period from 1997 to 2004, the first three PCS explain about 90-99% of the variance of rate changes.

Applications to Mortgage Securities

- **Applications to Mortgage Securities**
  - – First Stage: Estimation of Principal Components
  - – Second Stage: Estimation of Empirical PC Durations

Applications to Mortgage Securities

Second Stage: Estimation of Empirical PC Durations

- Now using the PC loadings obtained from 120 business days of data in Table 10.10, we compute the principal components for the following 30 business days.

- The PCs begins on January 13, 2004 until February 25, 2004 are shown in the last three columns of Table 10.11.

Table 10.11 PC Estimation Using the PC Loading Matrix

	Change in Rate							
	3-month rate	*1-year rate*	*2-year rate*	*5-year rate*	*10-year rate*	*PC(1)*	*PC(2)*	*PC(3)*
01/13/04	0.02	-0.039	-0.049	-0.07	-0.056	-0.108	0.005	-0.006
01/14/04	-0.01	0.015	0.016	-0.017	-0.044	-0.022	-0.046	-0.011
01/15/04	0	0.015	0.032	0.017	-0.011	0.025	-0.028	-0.014
01/16/04	0.01	0.025	0.023	0.063	0.051	0.085	0.016	0.012
…	…	…	…	…	…	…	…	…
02/25/04	-0.01	-0.019	-0.033	-0.021	-0.012	-0.042	0.016	0.004

Applications to Mortgage Securities

Second Stage: Estimation of Empirical PC Durations

- The time series of daily returns on any fixed income security can be now regressed against the corresponding times series of daily PC values given in Table 10.11 in order to get empirical PC durations.

- Assumptions:
  - Daily return on MBS, mortgage pass-through securities
  - Most trades are done on a to-be-announced basis
  - Buyers and sellers agree on general parameters
  - The trades satisfy the Good Delivery guidelines

Applications to Mortgage Securities

Second Stage: Estimation of Empirical PC Durations

- Table 10.12 shows the prices TBA FNMA pass-throughs with eight different coupons ranging from 5.5% to 9%, beginning Jan. 13 2004, until Feb. 25 2004.

	5.50%	*6%*	*6.50%*	*7%*	*7.50%*	*8%*	*8.50%*	*9%*
02/13/04	102.094	103.906	105.188	106.125	107.125	108	108	108.25
02/17/04	102.063	103.906	105.188	106.125	107.125	108	108	108.25
02/18/04	102.031	103.875	105.156	106.094	107.094	108	108	108.25
02/19/04	102.094	103.906	105.156	106.125	107.125	108	108	108.25
02/20/04	101.938	103.813	105.094	106.125	107.125	108	108	108.25
02/23/04	102.063	103.875	105.094	106.125	107.125	108	108	108.25
02/24/04	102.156	103.969	105.125	106.125	107.125	108	108	108.25
02/25/04	102.219	104	105.125	106.125	107.125	108	108	108.25

Applications to Mortgage Securities

Second Stage: Estimation of Empirical PC Durations

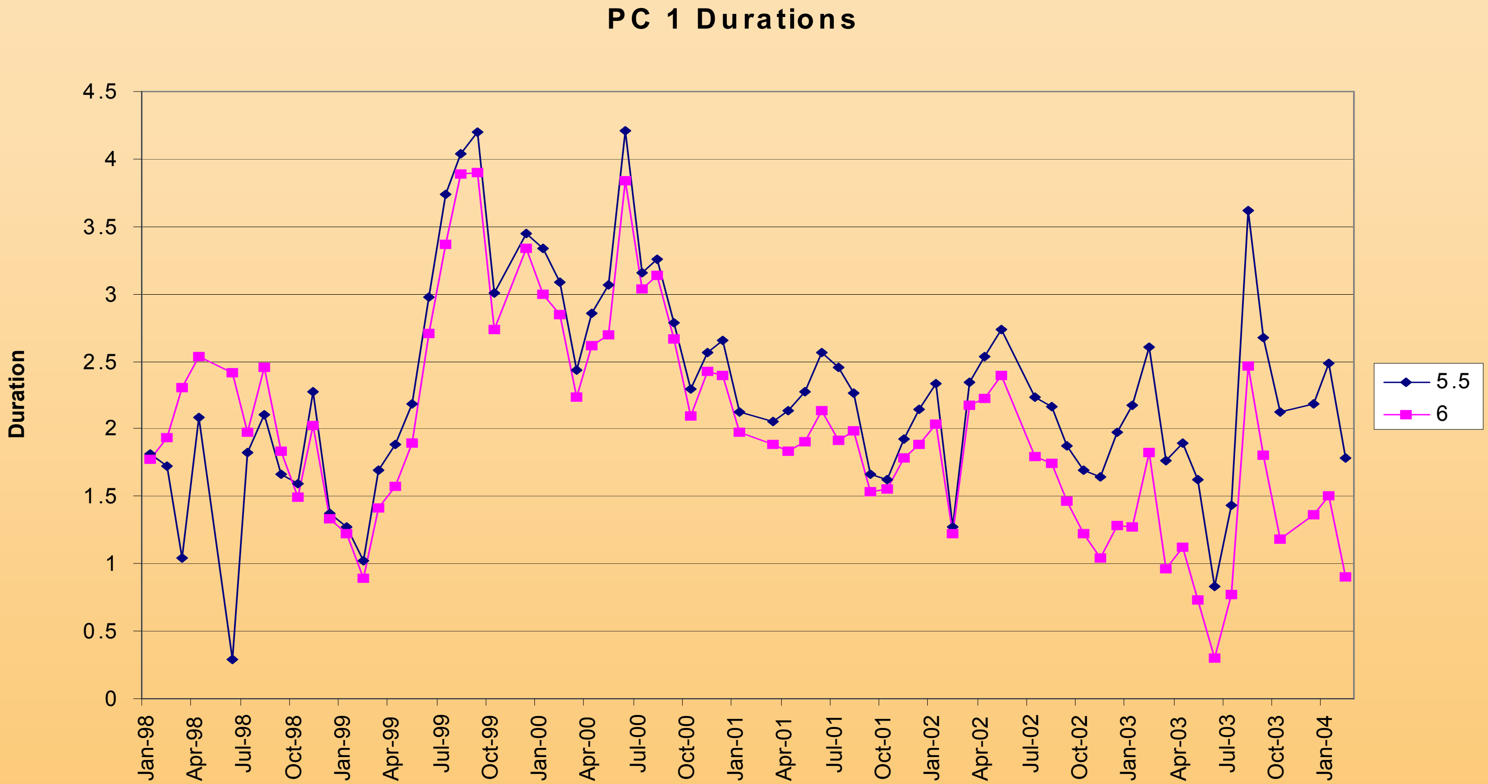
- The daily returns on the TBA FNMA pass-through with 5.5% coupon, together with the three PCs are shown in Table 10.13 as follows:

	Daily return on 5.5% coupon	*PC(1)*	*PC(2)*	*PC(3)*
01/13/04	0.15%	-0.108	0.005	-0.006
01/14/04	0.09%	-0.022	-0.046	-0.011
01/15/04	0.09%	0.025	-0.028	-0.014
01/16/04	-0.18%	0.085	0.016	0.012
…	…	…	…	…
02/25/04	0.06%	-0.042	0.016	0.004

Applications to Mortgage Securities

Second Stage: Estimation of Empirical PC Durations

- The empirical durations corresponding to the **first PC** for TBA FNMA pass-through with 5.5% coupon and 6% coupon are shown in Figure 10.7 as follows:



Applications to Mortgage Securities

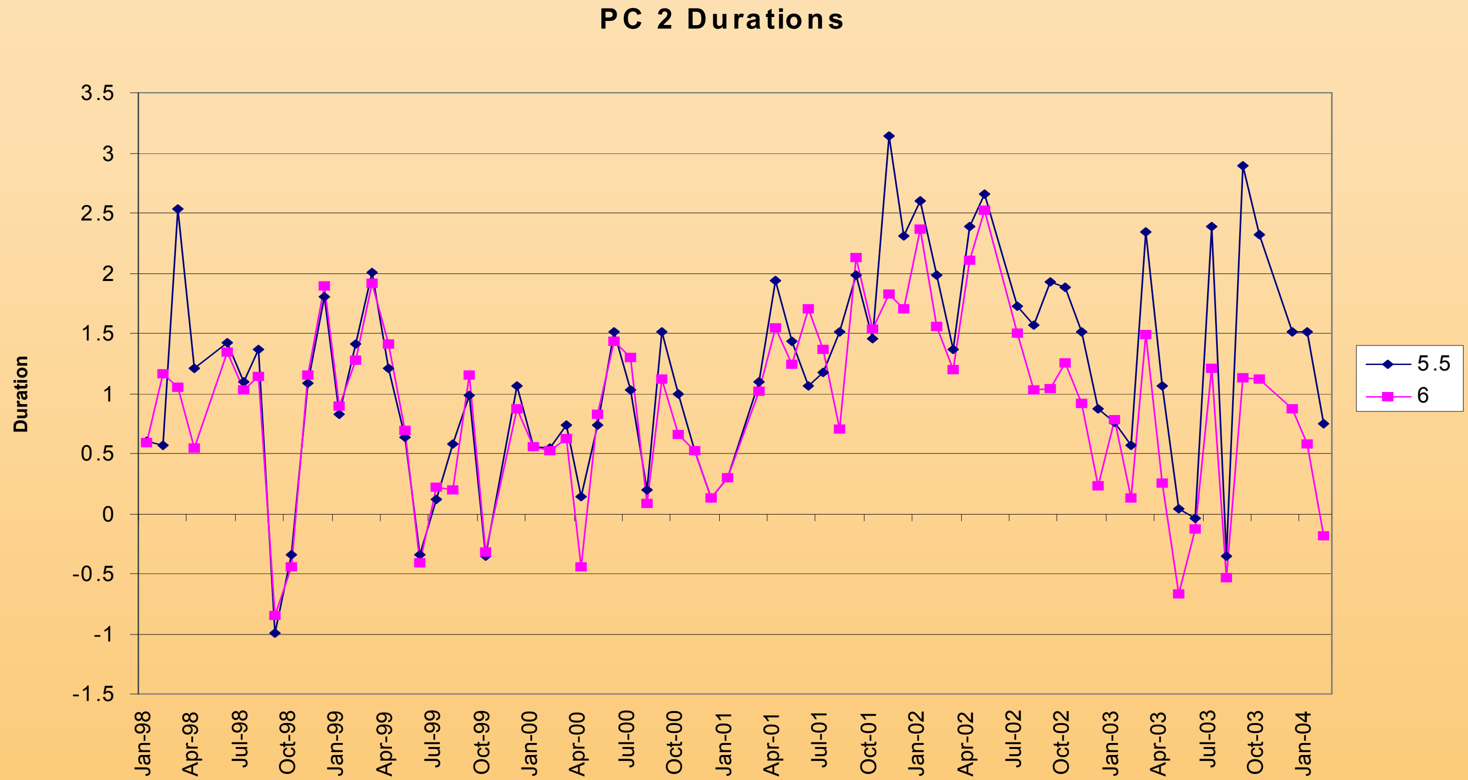
Second Stage: Estimation of Empirical PC Durations

- These durations have a similar pattern as was found for the duration of the Lehman MBS index in Figure 10.5.
- And due to higher coupon, the duration of the 6% pass-through is generally lower than that of the 5.5% pass-through.

Applications to Mortgage Securities

Second Stage: Estimation of Empirical PC Durations

- The empirical durations corresponding to the **second PC** for TBA FNMA pass-through with 5.5% coupon and 6% coupon are shown in Figure 10.8 as follows:



Applications to Mortgage Securities

Second Stage: Estimation of Empirical PC Durations

- These durations measure the sensitivity of these securities to slope shift in the Treasury yield curve.
- Though mostly the durations corresponding to the second PC are positive, implying a loss in value when positive slope shifts in the yield curve occur, these durations can become negative in some periods like July-November 1998 and in April-May 2003.

Interest Rate Risk Modeling
The Fixed Income Valuation Course

Sanjay K. Nawalkha
Gloria M. Soto
Natalia A. Beliaeva